Verification of the VirtuCath Pushability Simulation

The push and buckling solver checked against Euler's exact elastica, stepped columns, a refinement study, and contact and friction checks

VirtuCath LLC | VR-VC-005 | Version 1.2 | November 2026

Summary

VirtuCath's Pushability simulation pushes the free distal length of a shaft against a tip restraint and predicts the push force, the buckling knee, the buckled shape and any wall contact. It takes the stiffness of each shaft section and finds the equilibrium shape at every step of the stroke. This report checks that the solver behind it solves its equations correctly. The section stiffnesses it uses are validated separately, in *Validation of the VirtuCath Stiffness Model*.

0.13 %
largest push-force error against the exact elastica, default mesh

1.4 %
largest buckling-knee error against the Euler load, default starting bow

1.8
observed order of convergence: the error falls with the square of the element size

2,682
solved steps checked; every one converged and balanced

- **Exact solutions:** for a uniform column under each of the four ideal tip restraints, the push force after buckling matches Euler's exact elastica to within 0.13 % at the default mesh of 50 elements, and the buckled shape to within 0.073 %, starting from a near-perfect bow. Two-step columns, with one half four times stiffer than the other, match to within 0.16 %.
- **Buckling knee:** at the default starting bow, the knee of the force–stroke curve is within 1.4 % of the Euler load for every tip restraint.
- **Numerical error:** the force error falls with the square of the element size. Its median changes by at most 0.021 % with the number of stroke steps and 0.0088 % with the stiffness of the tip contact.
- **Contact and friction:** a buckling shaft first touches a channel wall where the exact shape reaches the wall, and the tip friction stays at the Coulomb limit while the tip slides.
- **Not yet done:** a comparison of predicted push forces with physical catheters (Section 5).

> **In short.** The pushability solver solves its mechanics correctly. Its remaining error, at most 0.13 % in push force at the default mesh, is a small, systematic discretization error, far below the uncertainty of the section stiffnesses it is given.

1 The pushability solver

The simulation clamps the shaft at the proximal fixture and pushes its free distal length, in stroke steps, against a tip restraint (Fig. 1): a frictionless wall the tip slides on (free tip), a captured tip that may rotate (pinned), a tip whose direction is held while it slides (guided), or a captured tip with its direction held (clamped). A physical wall mode adds an approach angle, a tip shape and friction, and the shaft can be pushed inside a straight channel or along a curved track. The shaft is a planar chain of short elements with the bending stiffness EI and the axial stiffness EA of each section. It starts from a small stress-free bow, with a tip offset of $L/500$ of the free length L by default, which sets the direction it buckles in.

At each stroke step, the solver computes the shaft's equilibrium shape and reports the push force at the clamp, the tip and wall forces and the shape.

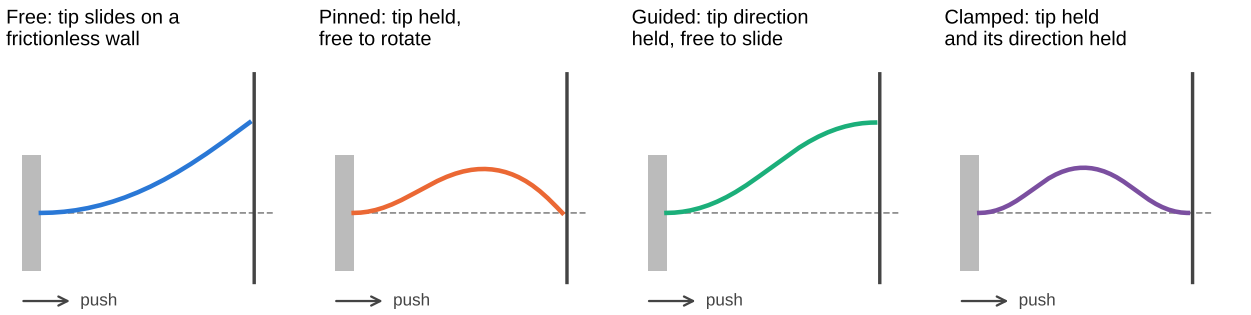

Figure 1. The four ideal tip restraints, with the exact buckled shape of a uniform column at an end shortening of 10 % of its length. The shaft is clamped at the left and pushed to the right against the tip restraint.

Intended use. Like the rest of VirtuCath, the Pushability simulation supports early and mid design: screening and comparing designs. VirtuCath does not replace design verification tests.

2 How the solver was checked

The checks follow ASME V&V 10 [1] and answer five questions (Table 1), from the code up to physical catheters.

Table 1. The evidence at a glance.

Question	How it was checked	Result	Status
1. Does the code solve its equations correctly?	Derivatives against finite differences; a free-body check on every solved step; behavior tests	All within tolerance	**Pass**
2. Does it reproduce exact solutions?	Euler loads and the exact post-buckling elastica for four tip restraints; two-step columns	Force within 0.13 %, knee within 1.4 %	**Pass**
3. Is the numerical error small?	Element count, starting bow, stroke steps and contact stiffness varied	Error falls with the square of the element size	**Pass**
4. Are contact and friction handled correctly?	First contact with a channel wall against the exact shape; tip friction against the Coulomb limit	Contact at the clearance to 1 μm; friction at the limit	**Pass**
5. Does it agree with physical catheters?	Measured push force on built shafts	–	Future work

3 Results

3.1 The code solves its equations correctly

The solver uses analytic derivatives of the shaft's energy, contact and friction included. On buckled states of every tip restraint, the physical wall and a two-section shaft in a frictional channel, they agree with finite differences of the energy to 8×10^{-11} of their largest entry. After every stroke step the solver also checks that the forces and moments on the shaft balance; all 2,682 solved steps in this report converged and passed that check. Further automated tests (Table 2) check basic behavior.

Table 2. Code verification. All tests pass on VirtuCath 1.7.1.

Check	Result	Criterion
Energy gradient against central differences (all restraints, wall, channel)	8×10^{-11}	10^{-5}
Net force and moments balance on every solved step (relative)	2,682 of 2,682	10^{-3}
The app's Euler reference line against the textbook Euler loads	within 0.000041 %	0.2 %
A square frictionless wall reproduces the free tip (relative)	holds	10^{-6}
A two-step column buckles between the two uniform columns	holds	–
A tilted low-friction wall lowers the force; a channel raises it	holds	–

3.2 It reproduces the exact elastica

A uniform column of length L and bending stiffness EI, clamped at the base, buckles at the Euler load $P_{cr} = \pi^2 EI/(KL)^2$, with $K = 2$, 0.699, 1 and 0.5 for a free, pinned, guided and clamped tip. Pushed further, it follows Euler's elastica: the push force rises slowly with the end shortening D, to 1.082 P_{cr} at $D = 0.15L$ for the free, guided and clamped tips and 1.057 P_{cr} for the pinned tip, and the buckled shape is known exactly. For the free, guided and clamped tips the shape is made of identical quarter waves and the solution has a closed form in elliptic integrals [2]. For all four, including the pinned tip, the same equations were also solved directly as a boundary-value problem [3]. The two routes agree to better than 10^{-6} where both apply.

The solver was run on a 50 mm column pushed through a stroke of 15 % of its length, with 25 to 200 elements and a near-perfect starting bow of $L/5000$, the smallest the app allows. Its push force was compared with the exact force at the same end shortening (Section 6).

Result. The solver follows the exact elastica closely for every tip restraint (Fig. 2, Table 3). At the default mesh of 50 elements, the push force after buckling is within 0.13 % of the exact value and the largest lateral offset of the shaft within 0.073 %. In almost every buckled step the solver's force is slightly below the exact force (99.7 % of buckled steps).

Table 3. Largest error of the push force against the exact elastica after buckling ($D > 0.02L$), by tip restraint and number of elements, with a starting bow of $L/5000$.

Tip restraint	25 elements	50	100	200
Free	0.13 %	0.11 %	0.10 %	0.12 %
Pinned	0.35 %	0.11 %	0.052 %	0.034 %
Guided	0.15 %	0.052 %	0.032 %	0.022 %
Clamped	0.53 %	0.13 %	0.035 %	0.012 %

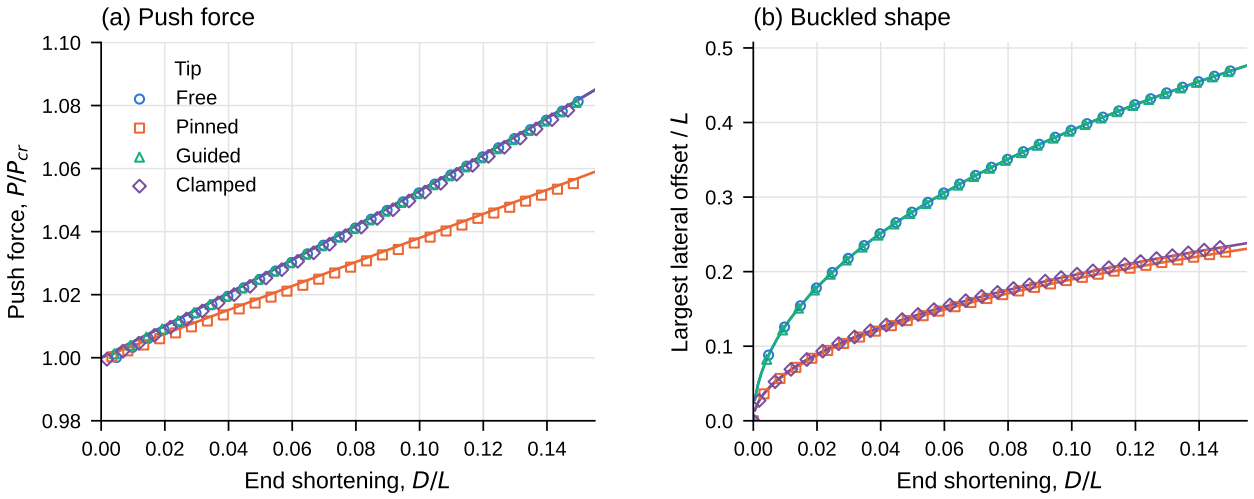

Figure 2. The solver (markers, 50 elements) against the exact elastica (lines): (a) push force and (b) largest lateral offset of the shaft, against the end shortening. The free, guided and clamped tips share one force curve when scaled by their own Euler load; the guided and free tips also share one shape curve.

Stepped columns. A shaft made of sections of different stiffness buckles at a load set by all of them. For a column of two equal halves, one four times stiffer than the other, the exact Euler load with a free tip is the root of a transcendental equation [2]; the app's Euler reference line matches it to a relative 6×10^{-9}. With the stiff half at the base or at the tip, and each of the four tip restraints, the solver's push force after buckling is within 0.16 % of the exact elastica at 50 elements (0.11 % at 100), and its knee within 0.38 % of the exact Euler load.

3.3 The numerical error is small

Elements. On the clamped tip, the force error falls with the square of the element size (Fig. 3), from 0.53 % at 25 elements to 0.012 % at 200, an observed order of 1.8. The pinned and guided tips start the same way and then level off, at 0.034 % and 0.022 % at 200 elements. The free tip's error stays at about 0.11 % at every mesh (Table 3): it is set by the starting bow, not the mesh (below). The app uses at least 50 elements, and more inside a channel or along a track, where the mesh is refined to the wall clearance and the tightest bend.

Starting bow and the buckling knee. A perfectly straight column has no preferred direction to buckle in, so the shaft starts from a small stress-free bow, a deliberate imperfection like that of a real shaft. It rounds the knee of the force–stroke curve and lowers the force just after buckling, most for the free tip, whose buckled shape it matches (Fig. 4). At 50 elements, the free tip's median force error is 0.059 % with a bow of $L/5000$, 0.53 % at the default $L/500$ and 1.3 % at $L/200$; for the other tips the largest error stays within 0.56 % at every bow. At the default bow the knee is within 1.4 % of the Euler load for every tip restraint, and no run stayed straight past the Euler load.

With the near-perfect bow of $L/5000$ the column also buckles at the Euler load. A nearly straight column pushed past that load still has a straight equilibrium, but it is unstable, and a solver that only looks for equilibrium can stop on it; the solver therefore checks every step that is still straight past the Euler load for stability and moves an unstable one onto the buckled shape. No run, uniform or stepped, stayed straight past the Euler load, the knee estimate was 1.00 to 1.00 P_{cr} at every mesh, and once buckled the force followed the elastica like the others (Table 3).

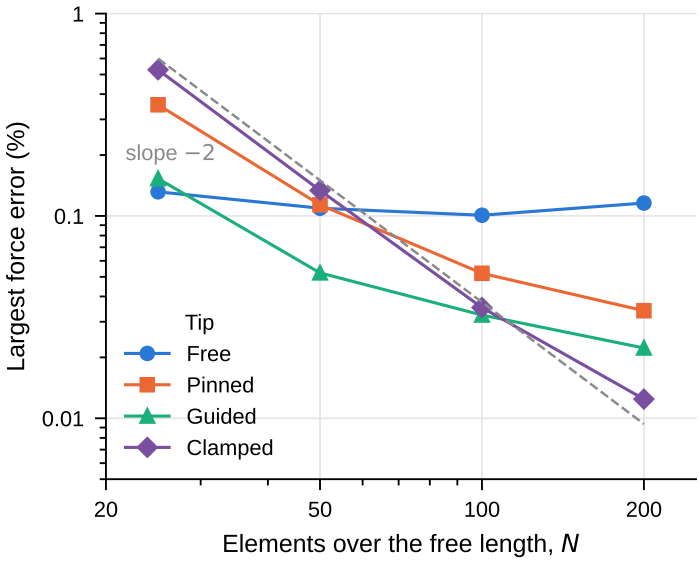

Figure 3. Largest push-force error against the exact elastica and the number of elements, starting bow $L/5000$. The dashed line has slope -2.

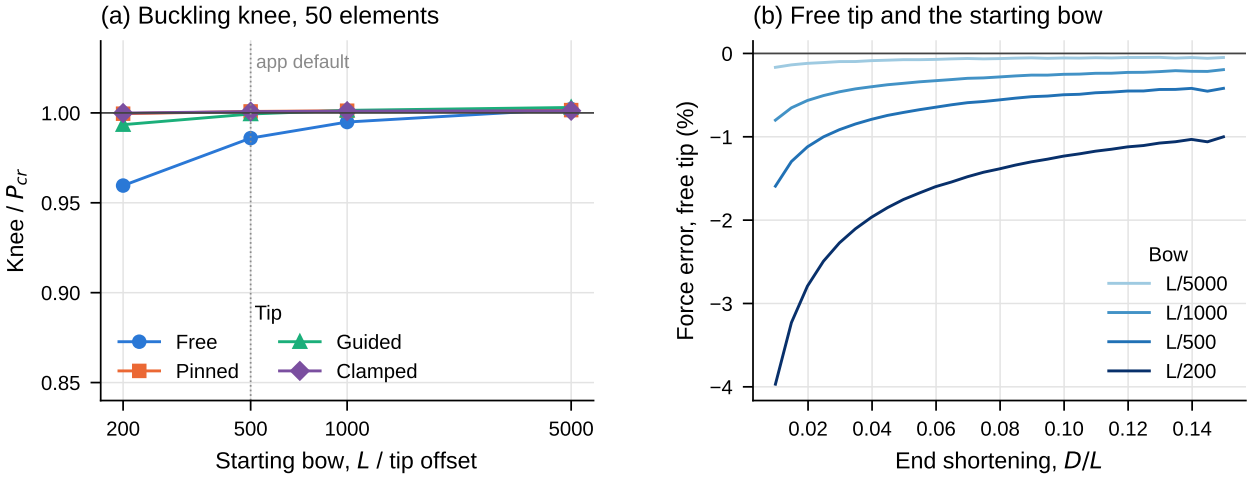

Figure 4. (a) The buckling knee, as a fraction of the Euler load, against the starting bow, at 50 elements. (b) The free tip's force error against the exact elastica, for four starting bows.

Stroke steps and contact stiffness. At the default bow, the median force error changed by no more than 0.021 % between 16 and 121 stroke steps, and by 0.0088 % when the stiffness of the tip contact was quartered or quadrupled. The contact stiffness sets how far the tip presses into its restraint (at most 0.17 % of the length at the default), which adds to the stroke but not to the end shortening. The knee, read against the stroke, stays within 0.986 to 1.003 P_{cr} over the step counts and 0.964 to 1.001 P_{cr} over the contact stiffnesses.

In short. Against exact solutions, the solver's error is a well-behaved discretization error: 0.13 % or less at the default mesh, falling with the square of the element size, and almost unchanged by the stroke steps and the contact stiffness. The default starting bow keeps the knee within 1.4 % of the Euler load.

3.4 Contact and friction are handled correctly

Channel walls. Inside a straight channel the shaft follows the exact elastica until it reaches a wall, so it should first touch the wall where the exact shape's largest lateral offset equals the clearance. For pinned and clamped tips and clearances of 2 and 4 mm per side (8 runs, 50 and 100 elements), the solver's shaft first touched the wall with its lateral offset equal to the clearance to within 1 μm, and within 72 μm of end shortening of the exact point, less than one stroke step (Fig. 5). After contact the wall supports the shaft and the push force rises, with drops where the shaft snaps into more waves, to 8 to 38 times the Euler load of the same tip restraint by the end of the stroke. That range has no exact solution and is covered by the free-body check.

Tip friction. On the physical wall, the tip friction follows Coulomb's law: while the tip slides, the friction force equals μ times the normal force, and while it sticks it stays below. Over friction coefficients μ of 0.1 to 0.6 and wall angles of 10° and 30°, the ratio of friction to normal force sat at the limit while the tip slid (median within 0.80 % of μ). Where the friction needed stayed below the limit, the tip stuck: with $\mu = 0.6$ at 10°, the ratio never exceeded 0.42. The largest excursion above the limit was 7.8 %, on stroke steps where the tip's normal force was falling quickly; the ratio returned to the limit as the normal force steadied.

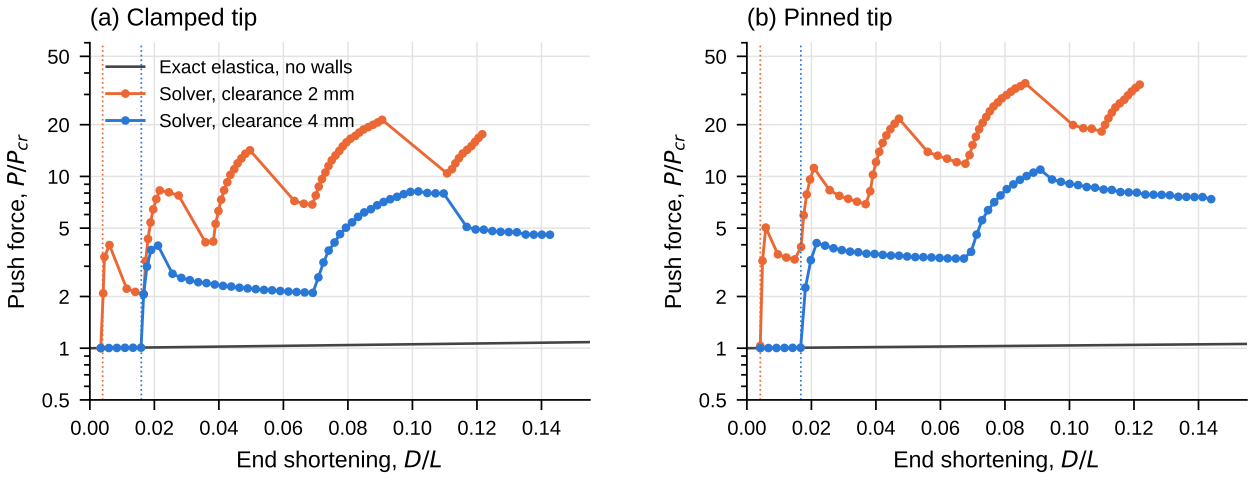

Figure 5. Push force in a straight channel, (a) clamped and (b) pinned tip, 100 elements, against the exact elastica without walls. Dotted lines: where the exact shape reaches the clearance. The force follows the elastica until the shaft touches the wall, then rises as the wall supports it, with drops where the shaft snaps into a new shape.

4 What this means for design work

Push force and knee. Under the ideal tip restraints, expect the push force within 0.13 % of the exact mechanics of the model at the default mesh, a median 0.53 % (at most 1.0 %) lower for the free tip at the default starting bow, and the knee within 1.4 % of the Euler load. These differences are small next to the uncertainty of the section stiffnesses, to which the Euler load is directly proportional, so the stiffness inputs, not the solver, set the accuracy of a push force for a real catheter.

Channels and walls. The onset of wall contact and the tip friction are solved correctly. Forces after contact depend on how the shaft settles against the walls, which this report checks through the free-body balance only.

Comparing designs. The solver's force sits slightly below the exact force in 99.7 % of buckled steps. An error with a consistent sign largely cancels when two similar designs are compared, so differences between designs and scenarios carry less of it than either absolute value.

5 Limits and next steps

Limits.

- **No physical validation yet.** The checks here compare the solver with exact solutions of the same idealized problem. They show the mechanics are solved correctly. They do not compare predicted push forces with physical catheters.
- **Planar model.** The shaft buckles in one plane. In a tube a real shaft can also take helical shapes, which the model does not represent.
- **After wall contact.** Push forces after a shaft meets a channel wall or a curved track have no exact reference here; they are covered by the free-body check only.
- **Friction.** The tip friction can exceed the Coulomb limit by up to 7.8 % on stroke steps where the normal force falls quickly.

Next step: bench validation of pushability. Push a small set of shafts with measured section stiffness against a load cell, under a clamped proximal fixture and the standard tip restraints, and in a channel or track, and compare the knee and the force–stroke curve with predictions made before the test.

6 Technical notes

Error metric. The push force is compared with the exact force at the same end shortening D of the shaft itself. This separates the elastica from the axial compression of the shaft and the small give of the tip contact, which add to the stroke. Errors are given as a percentage of the exact force, over the buckled range $D > 0.02L$.

Buckling knee. Straight lines are fitted to the start and the end of the force–stroke curve, before and after buckling. The knee is the point on the solved curve at the stroke where the two lines meet.

Data. The solver runs, the exact references and the scripts that produced every number and figure in this report are held by VirtuCath LLC and are available to reviewers on request.

Document history. Versions 1.0 and 1.1, October 2026, were issued as VR-VC-003 in error: that ID belongs to the stiffness model validation. Version 1.2, November 2026, corrects the document ID to VR-VC-005. The content of version 1.1 is unchanged.

Glossary. **Clearance**: the lateral distance the shaft's centerline can move before the shaft touches a channel wall, (channel diameter − shaft diameter)/2. **Elastica**: the exact large-deflection shape of an elastic rod. **End shortening**: how far the shaft's tip has moved toward its base. **Euler load**: the load at which a straight column buckles. **Knee**: the point where the push force stops rising steeply as the shaft buckles. **Starting bow**: the small stress-free curve the shaft starts from, given by its tip offset as a fraction of L. **Tip restraint**: how the distal tip is held (Fig. 1). **Verification**: is the model solved correctly? **Validation**: is it the right model for the use?

References

[1] ASME. V&V 10-2019: Standard for verification and validation in computational solid mechanics. American Society of Mechanical Engineers; 2019.
[2] Timoshenko SP, Gere JM. Theory of elastic stability. 2nd ed. New York: McGraw-Hill; 1961.
[3] Kierzenka J, Shampine LF. A BVP solver based on residual control and the MATLAB PSE. ACM Trans Math Softw. 2001;27:299–316.